3-SASAKIAN MANIFOLDS IN DIMENSION SEVEN, THEIR SPINORS AND G_2-STRUCTURES

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ABSTRACT. It is well-known that 7-dimensional 3-Sasakian manifolds carry a one-parametric family of compatible G_2-structures and that they do not admit a characteristic connection. In this note, we show that there is nevertheless a distinguished cocalibrated G_2-structure in this family whose characteristic connection ∇^c along with its parallel spinor field Ψ_0 can be used for a thorough investigation of the geometric properties of 7-dimensional 3-Sasakian manifolds. Many known and some new properties can be easily derived from the properties of ∇^c and of Ψ_0, yielding thus an appropriate substitute for the missing characteristic connection.

1. Introduction

3-Sasakian manifolds have been studied by the Japanese school in Differential Geometry decades ago [14]. They are Einstein spaces of positive scalar curvature carrying three compatible orthogonal Sasakian structures. In the middle of the 80-ties, a relation between 3-Sasakian manifolds and the spectrum of the Dirac operator was discovered [10], [11]. Indeed, they admit three Riemannian Killing spinors, which realize the lower bound for the eigenvalues of the Dirac operator [6]. Seven-dimensional, regular 3-Sasakian manifolds are classified in [10]. In the 90-ties, many new families of non-regular 3-Sasakian manifolds have been constructed specially in dimension seven [4]. This dimension is important because the exceptional Lie group G_2 admits a 7-dimensional representation and any 3-Sasakian-structure on a Riemannian manifold induces a family of adapted, non-integrable G_2-structures. A deformation of one of these G_2-structures—we call it the *canonical* G_2*-structure*—yields examples of 7-dimensional Riemannian manifolds with precisely one Killing spinor [12]. The whole family of underlying G_2-structures has been investigated from the viewpoint of spin geometry in [2], section 8. In particular, they are solutions of type II string theory with 4-fluxes (see [1] for more background and motivation).

We will show that the canonical G_2-structure of a 3-Sasakian manifold is cocalibrated. Consequently, there exists a unique connection with totally skew-symmetric torsion preserving it, see [8], [9]. The aim of this note is to study this characteristic connection ∇^c as well as the corresponding ∇^c-parallel spinor field Ψ_0. This point of view allows us to prove many properties of 3-Sasakian manifolds in a unified way. For example, the Riemannian Killing spinors are the Clifford products of the canonical spinor Ψ_0 by the three unit vectors defining the 3-Sasakian structure: in this sense, the ∇^c-parallel spinor field Ψ_0 is more fundamental than the Killing spinors. Finally we study the

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spinorial field equations and the deformations of the canonical G_2-structure in more detail.

2. 3-Sasakian manifolds in dimension seven

A 7-dimensional *Sasakian manifold* is a Riemannian manifold (M^7, g) equipped with a contact form η, its dual vector field ξ as well as with an endomorphism $\varphi : TM^7 \to TM^7$ such that the following conditions are satisfied:

$$\begin{aligned}
\eta \wedge (d\eta)^3 \neq 0, \quad \eta(\xi) \ &= \ 1, \quad g(\xi,\xi) \ = \ 1,\\
g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\cdot\eta(Y), \quad \varphi^2 \ &= \ -\mathrm{Id} \ + \ \eta\otimes\xi,\\
\nabla^g_X \xi = -\,\varphi X, \quad (\nabla^g_X \varphi)(Y) \ &= \ g(X,Y)\cdot\xi - \eta(Y)\cdot X.
\end{aligned}$$

These conditions imply several further relations, for example

$$\varphi\xi \ = \ 0, \quad \eta\circ\varphi \ = \ 0, \quad d\eta(X,Y) \ = \ 2\cdot g(X,\varphi Y).$$

A 7-dimensional 3-*Sasakian manifold* is a Riemannian manifold (M^7, g) equipped with three Sasakian structures $(\xi_\alpha, \eta_\alpha, \varphi_\alpha), \alpha = 1,2,3$, such that

$$[\xi_1\,,\,\xi_2] \ = \ 2\,\xi_3, \quad [\xi_2\,,\,\xi_3] \ = \ 2\,\xi_1, \quad [\xi_3\,,\,\xi_1] \ = \ 2\,\xi_2$$

and

$$\begin{aligned}
\varphi_3\circ\varphi_2 \ &= \ -\,\varphi_1 + \eta_2\otimes\xi_3, \quad & \varphi_2\circ\varphi_3 \ &= \ \varphi_1 + \eta_3\otimes\xi_2,\\
\varphi_1\circ\varphi_3 \ &= \ -\,\varphi_2 + \eta_3\otimes\xi_1, \quad & \varphi_3\circ\varphi_1 \ &= \ \varphi_2 + \eta_1\otimes\xi_3,\\
\varphi_2\circ\varphi_1 \ &= \ -\,\varphi_3 + \eta_1\otimes\xi_2, \quad & \varphi_1\circ\varphi_2 \ &= \ \varphi_3 + \eta_2\otimes\xi_1.
\end{aligned}$$

The vertical subbundle $\mathrm{T}^v \subset TM^7$ is spanned by ξ_1, ξ_2, ξ_3, its orthogonal complement is the horizontal subbundle T^h. Both subbundles are invariant under $\varphi_1, \varphi_2, \varphi_3$.

The properties as well as examples of Sasakian and 3-Sasakian manifolds are the topic of the book [4]. 3-Sasakian manifolds are always Einstein with scalar curvature $R = 42$. If they are complete, they are compact with finite fundamental group. Therefore we shall always assume that M^7 is compact and simply-connected. The frame bundle has a topological reduction to the subgroup $\mathrm{SU}(2) \subset \mathrm{SO}(7)$. In particular, M^7 is a spin manifold. Moreover, there exists locally an orthonormal frame $e_1, \ldots, e_7$ such that $e_1 = \xi_1$, $e_2 = \xi_2$, $e_3 = \xi_3$ and the endomorphisms φ_α acting on the horizontal part $\mathrm{T}^h := \mathrm{Lin}(e_4, e_5, e_6, e_7)$ of the tangent bundle are given by the following matrices

$$\varphi_1 := \begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{bmatrix}, \ \varphi_2 := \begin{bmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{bmatrix}, \ \varphi_3 := \begin{bmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We will identify vector fields with 1-forms via the Riemannian metric, thus obtaining a coframe $\eta_1, \eta_2, \ldots, \eta_7$, and shall use throughout the abbreviation $\eta_{ij\ldots} := \eta_i \wedge \eta_j \wedge \ldots$. In this frame, we compute the differentials $d\eta_\alpha$,

$$\begin{aligned}
d\eta_1 \ &= \ -\,2\,(\eta_{23} + \eta_{45} + \eta_{67}),\\
d\eta_2 \ &= \ \,2\,(\eta_{13} - \eta_{46} + \eta_{57}),\\
d\eta_3 \ &= \ -\,2\,(\eta_{12} + \eta_{47} + \eta_{56}).
\end{aligned}$$

Each of the three Sasaki structures on M^7 admits a characteristic connection, i.e. a metric connection with antisymmetric torsion; however, this torsion is well-known to

be $\eta_i \wedge d\eta_i$ [8, Thm 8.2], and these do not coincide for $i = 1, 2, 3$. Thus, a 3-Sasakian manifold has no characteristic connection [1, §2.6].

3. The canonical G_2-structure of a 3-Sasakian manifold

Consider the following 3-forms,

$$F_1 \ := \ \eta_1 \wedge \eta_2 \wedge \eta_3\,, \quad F_2 \ := \ \frac{1}{2}\big(\eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3\big) + 3\eta_1 \wedge \eta_2 \wedge \eta_3.$$

Then

$$\omega \ := \ F_1 + F_2 \ = \ \eta_{123} - \eta_{145} - \eta_{167} - \eta_{246} + \eta_{257} - \eta_{347} - \eta_{356}$$

is a generic 3-form defined globally on M^7. It induces a G_2-structure on M^7.

Definition 3.1. The 3-form $\omega = F_1 + F_2$ is called the *canonical* G_2*-structure* of the 7-dimensional 3-Sasakian manifold.

We investigate now the type of this canonical G_2-structure from the point of view of G_2-geometry [5], [8]. It is basically described by the differential of the G_2-structure ω. We compute directly [12]

$$dF_1 \ = \ 2 \cdot (*F_2), \quad dF_2 \ = \ 12 \cdot (*F_1) + 2 \cdot (*F_2)\,, \quad d * F_1 \ = \ d * F_2 \ = \ 0.$$

In particular, the canonical G_2-structure is cocalibrated. Equivalently, it is of type $\mathcal{W}_1 \oplus \mathcal{W}_3 = \Lambda^3_1 \oplus \Lambda^3_{27}$ in the Fernandez/Gray notation, see [5], [8], [9],

$$d * \omega \ = \ 0, \quad *d\omega \ = \ 4\,(3\,F_1 + F_2).$$

There exists a unique connection ∇^c preserving the G_2-structure with totally skew-symmetric torsion T^c [8], [9]. For a cocalibrated G_2-structure ω this *characteristic torsion form* T^c is given by the formula

$$\mathrm{T}^c \ = \ - * d\omega + \frac{1}{6}(d\omega,\ *\omega) \cdot \omega.$$

We express the characteristic torsion by the data of the 3-Sasakian structure,

$$\mathrm{T}^c \ = \ -6F_1 + 2F_2 \ = \ \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3 \ = \ 2\omega - 8\,F_1.$$

Thus, we see that T^c is the sum of the three characteristic torsion forms of the Sasakian structures η_i.

Let us decompose the characteristic torsion $\mathrm{T}^c = \mathrm{T}^c_1 + \mathrm{T}^c_{27}$ into the $\mathcal{W}_1 = \Lambda^3_1$- and the $\mathcal{W}_3 = \Lambda^3_{27}$ -part , respectively. Then we obtain

$$\mathrm{T}^c_1 \ = \ \frac{6}{7}\,(F_1 \ + \ F_2) \ = \ \frac{6}{7}\,\omega, \quad \mathrm{T}^c_{27} \ = \ \frac{8}{7}\,(F_2 \ - \ 6\,F_1).$$

In particular, the canonical G_2-structure of a 3-Sasakian manifold is never of pure type $\mathcal{W}_1$ or $\mathcal{W}_3$.

We will now prove that the canonical G_2-structure has parallel characteristic torsion, $\nabla^c\mathrm{T}^c = 0$, and realizes one type of cocalibrated G_2-structures with characteristic holonomy contained in the maximal, six-dimensional subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$ of $\mathfrak{g}_2$ [7]. Later, we shall see that its holonomy algebra coincides with $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$.

Theorem 3.1. *The canonical* G_2*-structure* ω *of a* 7*-dimensional* 3*-Sasakian manifold is cocalibrated,* $d * \omega = 0$*. Its characteristic torsion is given by the formula*

$$T^c \;=\; - * d\omega + 6\omega.$$

Moreover, we have $(d\omega, *\omega) = 36$*,* $|T^c|^2 = 60$ *and*

$$d * T^c \;=\; 0, \quad dT^c \;=\; -4 * T^c, \quad d\omega \;=\; \frac{1}{2}\, d * d\omega \;-\; 12 * \omega.$$

The characteristic connection preserves the splitting $TM^7 = T^v \oplus T^h$ *and the characteristic torsion is* ∇^c*-parallel,* $\nabla^c T^c = 0$*.*

Proof. Since ξ_1 is a Killing vector field, we have

$$\nabla^g_X \eta_1 \;=\; \frac{1}{2}\, X \lrcorner\, d\eta_1 .$$

Then we obtain

$$\nabla^c_X \eta_1 \;=\; \nabla^g_X \eta_1 \,+\, \frac{1}{2}\, T^c(X, \eta_1, -) \;=\; \frac{1}{2}\, X \lrcorner\, d\eta_1 \;-\; \frac{1}{2}\, X \lrcorner\, (\eta_1 \lrcorner\, T^c) \; .$$

The formula $T^c = \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3$ yields directly

$$\eta_1 \lrcorner\, T^c \;=\; d\eta_1 + (\eta_1 \lrcorner\, d\eta_2) \wedge \eta_2 + (\eta_1 \lrcorner\, d\eta_3) \wedge \eta_3 .$$

Moreover, the formulas for the differential $d\eta_\alpha$ imply that

$$\eta_1 \lrcorner\, d\eta_2 \;=\; 2\, \eta_3, \quad \eta_1 \lrcorner\, d\eta_3 \;=\; -\, 2\, \eta_2$$

holds. Thus we obtain

$$\nabla^c_X \eta_1 \;=\; 2\, X \lrcorner\, (\eta_2 \wedge \eta_3),$$

i. e. ∇^c preserves the subbundle T^v. Finally we have

$$(\nabla^c_X \eta_1) \wedge \eta_2 \wedge \eta_3 \;=\; 0$$

and then $\nabla^c(\eta_1 \wedge \eta_2 \wedge \eta_3) = 0$. Since $T^c \;=\; 2\omega - 8\, \eta_1 \wedge \eta_2 \wedge \eta_3$ and $\nabla^c \omega = 0$ we conclude that $\nabla^c T^c = 0$ holds, too. □

4. The canonical spinor of a 3-Sasakian manifold

Since the spin representation of Spin(7) is real, let us consider the real spinor bundle Σ. Any G_2-structure ω acts via the Clifford multiplication on Σ as a symmetric endomorphism with eigenvalue (-7) of multiplicity one and eigenvalue 1 of multiplicity seven. Consequently, any G_2-structure on a simply-connected manifold M^7 defines a *canonical spinor field* Ψ_0 such that (see [12], [8])

$$\omega \cdot \Psi_0 \;=\; -\, 7\, \Psi_0 \, , \quad |\Psi_0| \;=\; 1 \; .$$

If (M^7, ω) is cocalibrated and ∇^c is its characteristic connection , we obtain [8], [3]

$$\nabla^c \Psi_0 \;=\; 0\, , \quad T^c \cdot \Psi_0 \;=\; -\, \frac{1}{6}\, (d\omega, *\omega) \cdot \Psi_0\, , \quad \mathrm{Scal}^g \;=\; \frac{1}{18}\, (d\omega, *\omega)^2 \;-\; \frac{1}{2}\, |T^c|^2 \; ,$$

We apply the general formulas to the canonical spinor of a 3-Sasakian manifold M^7. Then we obtain a spinor field such that

$$\omega \cdot \Psi_0 \;=\; -\, 7\, \Psi_0\, , \quad T^c \cdot \Psi_0 \;=\; -\, 6\, \Psi_0\, , \quad \nabla^g_X \Psi_0 + \frac{1}{4}\, (X \lrcorner\, T^c) \cdot \Psi_0 \;=\; 0 \; .$$

Using the explicit formulas for ω and T^c, a direct algebraic computation in the real spin representation yields the following

Lemma 4.1.

$$\begin{aligned} \mathrm{T}^c \cdot X \cdot \Psi_0 &= -\frac{5}{3} X \cdot \mathrm{T}^c \cdot \Psi_0 \;=\; 10\, X \cdot \Psi_0 \quad \textit{if} \quad X \in \mathrm{T}^v, \\ \mathrm{T}^c \cdot X \cdot \Psi_0 &= X \cdot \mathrm{T}^c \cdot \Psi_0 \;=\; -6\, X \cdot \Psi_0 \quad \textit{if} \quad X \in \mathrm{T}^h, \end{aligned}$$

The equation $\nabla^c \Psi_0 = 0$ can be written as

$$\nabla^g_X \Psi_0 - \frac{1}{8}\,(X \cdot \mathrm{T}^c + \mathrm{T}^c \cdot X) \cdot \Psi_0 \;=\; 0\ .$$

We apply now the algebraic Lemma and obtain a differential equation involving the canonical spinor of a 3-Sasakian manifold.

Theorem 4.1. *The canonical spinor field Ψ_0 of a 7-dimensional 3-Sasakian manifold satisfies the following differential equation:*

$$\nabla^g_X \Psi_0 \;=\; \frac{1}{2}\, X \cdot \Psi_0 \quad \text{if} \quad X \in \mathrm{T}^v\,, \qquad \nabla^g_X \Psi_0 \;=\; -\frac{3}{2}\, X \cdot \Psi_0 \quad \text{if} \quad X \in \mathrm{T}^h\ .$$

In particular, Ψ_0 is an eigenspinor for the Riemannian Dirac operator, $D^g \Psi_0 = \frac{9}{2}\, \Psi_0$.

Remark 4.1. This equation has already been discussed in [7], section 10. It follows essentially from the formula $\mathrm{T}^c = 2\,\omega - 8\,F_1$.

5. ∇^c-parallel vectors and spinors of the canonical G_2-structure

The spinor bundle splits into three subbundles, $\Sigma = \Sigma_1 \oplus \Sigma_3 \oplus \Sigma_4$, where

$$\Sigma_1 \;:=\; \mathbb{R} \cdot \Psi_0, \quad \Sigma_3 \;:=\; \big\{ X \cdot \Psi_0 \,:\, X \in \mathrm{T}^v \big\}, \quad \Sigma_4 \;:=\; \big\{ X \cdot \Psi_0 \,:\, X \in \mathrm{T}^h \big\}.$$

The characteristic connection preserves this splitting. Obviously, the 3-form ω acts as the identity on $\Sigma_3 \oplus \Sigma_4$, while the torsion form satisfies

Lemma 5.1. *The torsion form T^c acts on Σ_3 as a multiplication by 10 and it acts on $\Sigma_1 \oplus \Sigma_4$ as a multiplication by (-6).*

Given the definition of Σ_4, it is now a crucial observation that ∇^c-parallel vector fields cannot be horizontal:

Proposition 5.1. *Horizontal, ∇^c-parallel vector fields*

$$\nabla^c X \;=\; 0, \quad 0 \;\neq\; X \in \Gamma(\mathrm{T}^c)$$

do not exist.

Proof. Let $0 \neq X$ be the vector field. Then $\Psi := X \cdot \Psi_0$ is a ∇^c-parallel spinor, too. Moreover, the torsion form acts on Ψ_0 and on Ψ by the same eigenvalue,

$$\mathrm{T}^c \cdot \Psi_0 \;=\; -6\, \Psi_0\,, \quad \mathrm{T}^c \cdot \Psi \;=\; -6\, \Psi\ .$$

The holonomy algebra $\mathfrak{hol}(\nabla^c)$ is contained in $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \subset \mathfrak{g}_2 \subset \mathfrak{so}(7)$ and the linear holonomy representation splits into $\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3$. The vector field X is an element of $\mathbb{R}^4$ such that $\mathfrak{hol}(\nabla^c) \cdot X = 0$. In [7] we explicitely realized the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$ inside $\mathfrak{so}(7)$. Using these formulas, an easy computation yields that the holonomy algebra is contained in $\mathfrak{so}(3) \subset \mathfrak{su}(3) \subset \mathfrak{g}_2$ and the linear holonomy representation splits into $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^1$. Consequently, the G_2-manifold (M^7, ω) is cocalibrated, its characteristic holonomy is contained in $\mathfrak{so}(3)$ and the characteristic torsion T^c acts on both ∇^c-parallel spinors with the same eigenvalue. It turns out that M^7 cannot be an Einstein manifold with positive scalar curvature by [7, Thm 7.1], a contradiction. $\square$

In general, the Casimir operator of a metric connection with parallel characteristic torsion is given by the following formulas [3]

$$\Omega \;=\; (D^{1/3})^2 - \frac{1}{16}(2\,\mathrm{Scal}^g + |\mathrm{T}^c|^2) \;=\; \Delta_{\mathrm{T}^c} + \frac{1}{16}\,(2\,\mathrm{Scal}^g + |\mathrm{T}^c|^2) - \frac{1}{4}\,(\mathrm{T}^c)^2.$$

Its kernel contains the space of all ∇^c-parallel spinor fields. In particular, any ∇^c-parallel spinor field Ψ satisfies the algebraic condition [8], [3]

$$4\,(\mathrm{T}^c)^2 \cdot \Psi \;=\; (2\,\mathrm{Scal}^g \;+\; |\mathrm{T}^c|^2) \cdot \Psi.$$

For the canonical G_2-structure of a 3-Sasakian manifold we have $2\,\mathrm{Scal}^g + |\mathrm{T}^c|^2 = 144$. Consequently, any ∇^c-parallel spinor field is a section in the subbundle $\Sigma_1 \oplus \Sigma_4$, i.e. of the form $\Psi = a \cdot \Psi_0 + X \cdot \Psi_0$, where a is constant and $X \in \Gamma(\mathrm{T}^h)$ is a horizontal, parallel vector field. But horizontal, ∇^c-parallel vector fields do not exist. This argument proves:

Theorem 5.1. *Any ∇^c-parallel spinor field is proportional to Ψ_0. Moreover, the holonomy algebra is the six-dimensional maximal subalgebra $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$ of $\mathfrak{g}_2$.*

The latter argument proves that vertical, ∇^c-parallel vector fields do not exist. Indeed, if $\nabla^c X = 0$, then $X \cdot \Psi_0 \in \Gamma(\Sigma_3)$ is a parallel spinor in Σ_3. We conclude that $X \cdot \Psi_0 = 0$ and $X = 0$. Together with Proposition 5.1 and the splitting of the tangent bundle, one concludes:

Theorem 5.2. *There are no non-trivial ∇^c-parallel vector fields.*

6. Riemannian Killing spinors on 3-Sasakian manifolds

Consider the spinor fields $\Psi_1 := \xi_1 \cdot \Psi_0$, $\Psi_2 := \xi_2 \cdot \Psi_0$, $\Psi_3 := \xi_3 \cdot \Psi_0$. These spinors are sections in the bundle Σ_3.

Theorem 6.1. *The spinor fields Ψ_α are Riemannian Killing spinors, i.e.*

$$\nabla^g_X \Psi_\alpha \;=\; \frac{1}{2}\, X \cdot \Psi_\alpha\,, \quad \alpha = 1, 2, 3.$$

Corollary 6.1 ([10], [11])**.** *Any simply-connected 3-Sasakian manifold admits at least three Riemannian Killing spinors.*

Proof. We use the differential equation

$$\nabla^g_X \Psi_0 \;=\; \frac{1}{8}\,(X \cdot \mathrm{T}^c + \mathrm{T}^c \cdot X) \cdot \Psi_0$$

as well as the properties of Sasakian structures. Then we obtain

$$\begin{aligned}
\nabla^g_X(\xi_1 \cdot \Psi_0) \;&=\; \big(\nabla^g_X \xi_1\big) \cdot \Psi_0 + \xi_1 \cdot \nabla^g_X \Psi_0 \\
&=\; -\varphi_1(X) \cdot \Psi_0 + \frac{1}{8}\,\xi_1 \cdot (X \cdot \mathrm{T}^c + \mathrm{T}^c \cdot X) \cdot \Psi_0 \\
&=\; \frac{1}{2}\,\big(X \lrcorner\, d\eta_1\big) \cdot \Psi_0 + \frac{1}{8}\,\xi_1 \cdot (X \cdot \mathrm{T}^c + \mathrm{T}^c \cdot X) \cdot \Psi_0 \\
&=\; -\frac{1}{4}\big(X \cdot d\eta_1 - d\eta_1 \cdot X\big) \cdot \Psi_0 + \frac{1}{8}\,\xi_1 \cdot (X \cdot \mathrm{T}^c + \mathrm{T}^c \cdot X) \cdot \Psi_0.
\end{aligned}$$

A direct algebraic computation yields now that

$$-\frac{1}{4}\big(X \cdot d\eta_1 - d\eta_1 \cdot X\big) \cdot \Psi_0 + \frac{1}{8}\,\xi_1 \cdot (X \cdot \mathrm{T}^c + \mathrm{T}^c \cdot X) \cdot \Psi_0 \;=\; \frac{1}{2}\,X \cdot \xi_1 \cdot \Psi_0$$

holds specially for the spinor Ψ_0. This proves the statement of the Theorem. $\Box$

In general, any real spinor field Φ of length one defined on a 7-dimensional Riemannian manifold induces a G_2-structure ω_Φ (see [12]). Moreover, if two spinor fields $\Phi_2 = \xi \cdot \Phi_1$ are related via Clifford multiplication by some vector field ξ, then

$$\omega_{\Phi_2} \;=\; -\,\omega_{\Phi_1} + 2(\xi \,\lrcorner\, \omega_{\Phi_1}) \wedge \xi$$

holds [12, Remark 2.3]. Denote by ω_α the nearly parallel G_2-structure induced by the Riemannian Killing spinor $\Psi_\alpha = \xi_\alpha \cdot \Psi_0$ ($\alpha = 1, 2, 3$). Then we obtain

$$\omega_\alpha \;=\; -\frac{1}{2}\,(\eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3) - 4\,\eta_1 \wedge \eta_2 \wedge \eta_3 + 2(\xi_\alpha \,\lrcorner\, \omega) \wedge \eta_\alpha .$$

Consider, for example, the case $\alpha = 1$. Then

$$\xi_1 \,\lrcorner\, \omega \;=\; \frac{1}{2}\,d\eta_1 + \frac{1}{2}(\xi_1 \,\lrcorner\, d\eta_2) \wedge \eta_2 + \frac{1}{2}\,(\xi_1 \,\lrcorner\, d\eta_3) \wedge \eta_3 + 4\,\eta_{23} \;=\; \frac{1}{2}\,d\eta_1 + 2\,\eta_{23}.$$

Inserting the latter formula, we obtain

$$\begin{aligned} \omega_1 \;&=\; \frac{1}{2}\,\eta_1 \wedge d\eta_1 - \frac{1}{2}\,\eta_2 \wedge d\eta_2 - \frac{1}{2}\,\eta_3 \wedge d\eta_3 \\ &=\; \eta_{123} - \eta_{145} - \eta_{167} + \eta_{246} - \eta_{257} + \eta_{347} + \eta_{356}. \end{aligned}$$

Theorem 6.2. *The nearly parallel* G_2*-structures* $\omega_1, \omega_2, \omega_3$ *induced by the Killing spinors of a* 3*-Sasakian manifold are given by the formulas*

$$\begin{aligned} \omega_1 \;&=\; \frac{1}{2}\,\eta_1 \wedge d\eta_1 \,-\, \frac{1}{2}\,\eta_2 \wedge d\eta_2 \,-\, \frac{1}{2}\,\eta_3 \wedge d\eta_3 \\ \omega_2 \;&=\; -\,\frac{1}{2}\,\eta_1 \wedge d\eta_1 \,+\, \frac{1}{2}\,\eta_2 \wedge d\eta_2 \,-\, \frac{1}{2}\,\eta_3 \wedge d\eta_3 \\ \omega_3 \;&=\; -\,\frac{1}{2}\,\eta_1 \wedge d\eta_1 \,-\, \frac{1}{2}\,\eta_2 \wedge d\eta_2 \,+\, \frac{1}{2}\,\eta_3 \wedge d\eta_3 \;. \end{aligned}$$

All three nearly parallel G_2*-structures satisfy the equation* $d\omega_\alpha = -\,4\,(*\omega_\alpha)$.

Remark 6.1. The nearly parallel structures ω_α admit characteristic connections, too. Their characteristic torsions T^c_α are proportional to ω_α [8]. Moreover, the existence of a nearly parallel G_2-structure or—equivalently—of a Riemannian Killing spinor implies that M^7 is Einstein [6]. Consequently, our construction explains why 3-Sasakian manifolds are Einstein manifolds.

7. Deformations of the canonical G_2-structure

Deformations of 3-Sasakian metrics from the viewpoint of G_2-geometry have been studied in [12] and [7]. We once again describe the construction of these particular G_2-structures and their properties in a unified way, and add some more. Fix a positive parameter $s > 0$ and consider a new Riemannian metric g^s defined by

$$g^s(X,Y) \;:=\; g(X,Y) \quad \text{if} \quad X, Y \in \mathrm{T}^h, \quad g^s(X,Y) \;:=\; s^2 \cdot g(X,Y) \quad \text{if} \quad X, Y \in \mathrm{T}^v.$$

Then $s\eta_1, s\eta_2, s\eta_3, \eta_4, \ldots, \eta_7$ is an orthonormal coframe and we replace the 3-forms

$$\begin{aligned} F_1 \;&=\; \eta_1 \wedge \eta_2 \wedge \eta_3 \;, \\ F_2 \;&=\; \frac{1}{2}\big(\eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3\big) + 3\eta_1 \wedge \eta_2 \wedge \eta_3 \\ &=\; -\eta_{145} - \eta_{167} - \eta_{246} + \eta_{257} - \eta_{347} - \eta_{356} \end{aligned}$$

by the new forms

$$F_1^s \;:=\; s^3 F_1, \quad F_2^s \;:=\; sF_2, \quad \omega^s \;:=\; F_1^s + F_2^s.$$

(M^7, g^s, ω^s) is a Riemannian 7-manifold equipped with a G_2-structure ω^s. Denote by $*_s$ the corresponding Hodge operator acting on forms. We summarize some well-known properties of these G_2-structures that follow from a straightforward computation.

Theorem 7.1 ([12], Theorem 5.4 and [7], §10)**.**

(1) *The* G_2*-manifold* (M^7, g^s, ω^s) *is cocalibrated,* $d *_s \omega^s = 0$.

(2) *The differential of the* G_2*-structure is given by the formula*

$$d\omega^s \;=\; 12\, s\,(*_s F_1^s) + \big(2s + \frac{2}{s}\big)(*_s F_2^s).$$

(3) *The characteristic torsion* T^c_s *is given by the formula*

$$\mathrm{T}^c_s \;=\; \big(\frac{2}{s} - 10s\big)(s\eta_1) \wedge (s\eta_2) \wedge (s\eta_3) + 2s\omega^s.$$

(4) *The Riemannian Ricci tensor is given by the formula*

$$\mathrm{Ric}^{g^s} \;=\; 6\,(2 \,-\, s^2)\,\mathrm{Id}_{\mathrm{T}^h} \,\oplus\, \frac{2 + 4s^4}{s^2}\,\mathrm{Id}_{\mathrm{T}^v}.$$

In particular, the scalar curvature equals

$$\mathrm{Scal}^{g^s} \;=\; 6\,(8 + \frac{1}{s^2} - 2s^2).$$

(5) *The canonical spinor field* Ψ_0 *satisfies the differential equation*

$$\nabla^{g^s}_X \Psi_0 \;=\; -\frac{3}{2} s\, X \cdot \Psi_0 \quad \textit{if} \quad X \in \mathrm{T}^h,$$
$$\nabla^{g^s}_X \Psi_0 \;=\; \big(-\frac{1}{2s} + s\big)\, X \cdot \Psi_0 \quad \textit{if} \quad X \in \mathrm{T}^v.$$

Corollary 7.1 ([12], Theorem 5.4)**.** *For* $s = 1/\sqrt{5}$ *the* G_2*-structure is nearly parallel and* Ψ_0 *is a Riemannian Killing spinor,*

$$d\omega^s \;=\; \frac{12}{\sqrt{5}}\,(*_s\omega^s), \quad \mathrm{Ric}^{g^s} \;=\; \frac{54}{5}\,\mathrm{Id}, \quad \nabla^{g^s}_X \Psi_0 \;=\; -\frac{3}{2\sqrt{5}}\, X \cdot \Psi_0.$$

Ψ_0 *is the unique Riemannian Killing spinor of the metric.*

Remark 7.1. The Ricci tensor of the characteristic connection of (M^7, g^s, ω^s) is given by the formula [7]

$$\mathrm{Ric}^{\nabla^{c,s}} \;=\; 12\,(1 \,-\, s^2)\,\mathrm{Id}_{\mathrm{T}^h} \,\oplus\, 16\,(1 \,-\, 2\,s^2)\,\mathrm{Id}_{\mathrm{T}^v}.$$

If $s = 1$ (the 3-Sasakian case), then Ric^{∇^c} vanishes on the subbundle T^h. For $s = 1/\sqrt{5}$, the Ricci tensor is proportional to the metric, $\mathrm{Ric}^{\nabla^{c,1/\sqrt{5}}} = (48/5)\,\mathrm{Id}_{TM^7}$. From this point of view there is a third interesting parameter, namly $s^2 = 1/2$. Then the ∇^c-Ricci tensor vanishes on the subbundle T^v and the canonical spinor field Ψ_0 is parallel in vertical directions. It is a transversal Killing spinor with respect to the three-dimensional foliation and

$$\big(D^{g^s}\big)^2 \Psi_0 \;=\; 18\,\Psi_0 \;=\; \frac{1}{4}\frac{4}{4-1}\mathrm{Scal}^{g^s}\Psi_0.$$

In particular, Ψ_0 is the first known example to realize the lower bound for the basic Dirac operator of the foliation, see the recent work by Habib and Richardson [13].

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